May 2, 2000 (.)
UMN-TH-1841/00 , TPI-MINN-00/9-T
,OUTP-00-02-P

Gravitating Instantons In 3 Dimensions

Andrew Ferstl[a], Bayram Tekin[b,1], Victor Weir[a]

[a] *School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455*

[b] *Theoretical Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, UK*

Abstract

We study the Einstein-Chern-Simons gravity coupled to Yang-Mills-Higgs theory in three dimensional Euclidean space with cosmological constant. The classical equations reduce to Bogomol'nyi type first order equations in curved space. There are BPS type gauge theory instanton (monopole) solutions of finite action in a gravitational instanton which itself has a finite action. We also discuss gauge theory instantons in the vacuum (zero action) AdS space. In addition we point out to some exact solutions which are singular.

PACS: 04.20.-q , 04.60.-m
Keywords: Quantum Gravity, Instantons.

[1] e-mail: tekin@thphys.ox.ac.uk

1 Introduction

In recent years there has been a growing interest in the study of three dimensional gravity [1]. For example the BTZ [2] solution in three dimensions proved to be an extremely useful toy model to understand microscopic degrees of freedom of a black hole. Quite recently [3] a black hole with all three Abelian hairs (charge, angular momentum and mass) was shown to exist.

On the other hand, since the surprising numerical evidence of BK [4] on the existence of particle-like solutions (with hair) in four dimensional Einstein-Yang-Mills theory, there has been a lively research in the theories of gravity coupled to non-Abelian gauge theories. A nice account of developments and references in the subject is summarized in the review article [5]. More recently new monopole and dyon solutions were found in [6].

In this paper our intention is to study three dimensional Euclidean gravity coupled to Yang-Mills-Higgs theory. Deser's [7] earlier work in the study of Einstein-Yang-Mills theory (no Higgs) shows that there are no static solutions in this theory. In this paper we add a Higgs field in the adjoint representation of the group $SO(3)$ which is spontaneously broken down to $U(1)$ and we study the effect of gravity on the 't Hooft-Polyakov instantons in the BPS limit (the limit of vanishing self-interaction for the Higgs field). We show that one obtains the Bogomol'nyi type first order equations for the Higgs and the gauge field as in the flat space limit. There are exact solutions to the equations of motion which do not have flat space analogs (limits) but these solutions are not of finite action. We find the numerical solutions of finite action which reduce to the BPS instantons in flat space. The most interesting thing that we found is that gauge theory instantons exist in gravitational instantons. Gravitational instantons are not disturbed by the the gauge theory instantons.

The reader might wonder if a spontaneously broken, $SO(3)$ down to $U(1)$, gauge theory is expected to be any different from the Einstein-Maxwell theory in which a BTZ solution was found. In the four dimensional context it [8] was shown that, on the contrary to the initial expectation, the spontaneously broken gauge theory allowed black holes with "non-trivial" hair which do not exist in Einstein-Maxwell theory. So, in principle, we

do not have a strong reason to believe that the spontaneously broken theory, in three dimensions, will only have a BTZ type solution with only three types of hair. Although this question along with the question of non-Abelian hair of a BTZ black hole are extremely interesting we do not attempt these in this paper. Our immediate interest, as stated in the previous paragraph, is to explore what happens to the gauge theory instantons if gravity (with cosmological constant) is turned on. And more specifically we will explore what happens to the gauge theory instantons in the case that the space-time is a gravitational instanton. In section 2 we present the model in which we will search for an answer to this problem. In section 3 we will begin to solve the equations of motion, discuss some of the remaining symmetries, and show that the action does not depend on a specific choice of the coordinates. In section 4 we will find numerical solutions. In section 5 we will present an exact, singular solution to the equations of motion and finally in Section 6 we will make some concluding remarks.

2 The Model

We will work in the Euclidean space and in the first order formalism of gravity in terms of the dreibein and the spin connection. Achúcarro-Townsend [9] and Witten [10] showed that three dimensional Einstein-Hilbert action with zero cosmological constant is equivalent to Chern-Simons theory with the gauge group $ISO(3)$. In the theories with non-zero cosmological constant one can simply generalize this to $SO(4)$ or $SO(3,1)$ depending on the sign of the cosmological constant. Witten also realized that depending on the choice of the quadratic Killing-form one obtains two classically equivalent actions for gravity.

The standard action is the following [2]

$$S_G = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\epsilon^{ijk}\left\{2\, e^a{}_i\, \partial_j\, \omega^a{}_k + \epsilon^{abc}\, e^a{}_i\, \omega^b{}_j\, \omega^c{}_k + \frac{\lambda}{3}\epsilon^{abc}\, e^a{}_i\, e^b{}_j\, e^c{}_k\right\} \tag{1}$$

This action is real in the Euclidean space and is equivalent to Einstein-Hilbert theory if the dreibein is invertible. Another action, which in Minkowski space is equivalent to the

[2]In our conventions the integrand of the Euclidean path integral is e^{+S}.

Einstein-Hilbert action, often coined as "exotic action" is:

$$
\begin{aligned}
S_{CS} = \quad & -\frac{ik}{8\pi G}\int_{\mathcal{M}} d^3x\epsilon^{ijk}\Big\{\omega^a{}_i\,\partial_j\,\omega^a{}_k + \frac{1}{3}\epsilon^{abc}\,\omega^a{}_i\,\omega^b{}_j\,\omega^c{}_k + \lambda(e^a{}_i\partial_j\,e^a{}_k \\
& +\epsilon^{abc}\,\omega^a{}_i\,e^b{}_j\,e^c{}_k)\Big\}
\end{aligned} \tag{2}
$$

Assuming that the fields are real, which we do for our analysis, this action is completely imaginary because it is Wick rotated from Minkowski space. Both of these actions are considered in the path integral formulation of gravity. These actions are important geometric invariants of three manifolds; namely, they are the "volume" and the "Chern-Simons" invariants respectively. In this paper we consider both of them together. So including the Higgs and the Yang-Mills terms our full action becomes

$$
S = S_G + S_{CS} + S_{YM} + S_H \tag{3}
$$

where the Yang-Mills and the Higgs actions are

$$
S_{YM} = -\frac{1}{4e^2}\int d^3x\sqrt{|g|}g^{ij}g^{kl}F^a_{ik}F^a_{jl} \tag{4}
$$

$$
S_H = -\frac{1}{e^2}\int d^3x\sqrt{|g|}\{\frac{1}{2}g^{ij}D_ih^aD_jh^a + \frac{\nu}{6!}(h^2-h_0^2)^3\} \tag{5}
$$

The Higgs field is in the adjoint representation of $SO(3)$ and the covariant derivative is $D_ih^a = \partial_ih^a + \epsilon^{abc}A^b_ih^c$. Hence, F^a_{ij} has no gauge coupling in it.

Let us denote the dimensions of the fields and the parameters in the theory.

$$
\begin{aligned}
&[e^2] = M, \qquad [G] = M^{-1}, \qquad [\lambda] = M^2, \qquad [k] = M^{-1} \\
&[e^a{}_j] = M^0, \qquad [h] = M, \qquad [\omega^a{}_j] = M, \qquad [\nu] = M^{-2}
\end{aligned} \tag{6}
$$

We will exclusively work in the BPS limit where $\nu = 0$. The indices (a,b,c) denote the tangent space and (i,j,k) denote the manifold coordinates. The metrics , η_{ab} and g_{ij} have Euclidean signature. $\lambda < 0$ corresponds to the de-Sitter and $\lambda > 0$ to the anti-de-Sitter space. The "dual" Riemann tensor can be defined to be $R^a{}_{kj} = \partial_k\omega^a{}_j - \partial_j\omega^a{}_k + \epsilon^a{}_{bc}\omega^b{}_k\omega^c{}_j$.

The relation between the Ricci tensor and the dual Riemann tensor is $R_{ij} = e_i^a E_b^k \epsilon_{abc} R^c{}_{jk}$, where E_b^k is the inverse of the dreibein. In the absence of gauge fields Einsteins equations $R_{ij} = -2\lambda g_{ij}$ imply the scalar curvature to be $R = -6\lambda$.

We employ the well known spherically symmetric ansatz for all the fields in the theory.

$$e^a{}_j(\vec{x}) = \frac{G}{r}\left[-\epsilon^a{}_{jk}\,\hat{x}^k\,\phi_1 + \delta^a{}_j\,\phi_2 + (rA - \phi_2)\,\hat{x}^a\hat{x}_j\right] \tag{7}$$

$$w^a{}_j(\vec{x}) = \frac{1}{r}\left[\epsilon^a{}_{jk}\,\hat{x}^k\,(1-\psi_1) + \delta^a{}_j\psi_2 + (rB - \psi_2)\hat{x}^a\hat{x}_j\right] \tag{8}$$

$$A_j^a(\vec{x}) = \frac{1}{r}\left[\epsilon^a{}_{jk}\,\hat{x}^k\,(1-\varphi_1) + \delta^a{}_j\varphi_2 + (rD - \varphi_2)\hat{x}^a\hat{x}_j\right] \tag{9}$$

$$h^a(\vec{x}) = \hat{x}^a h(r) \tag{10}$$

The functions A, B, ϕ_α ,D, φ_α and ψ_α depend on r only. The meaning of r should be clear from $r^2 = \eta_{ij}x^i x^j$ and we define $\hat{x}^j = x^j/r$. We have chosen the dreibein to be dimensionless and the first term of the dreibein is chosen in a way which will yield more transparent equations.

The metric on the manifold can be recovered from the dreibein through the relation $g_{ij} = \eta_{ab}e^a{}_i e^b{}_j$ which yields;

$$g_{ij} = \frac{G^2}{r^2}\left\{(\phi_1^2 + \phi_2^2)(\delta_{ij} - \hat{x}_i\hat{x}_j) + r^2A^2\hat{x}_i\hat{x}_j\right\} \tag{11}$$

The flat space limit $(g_{ij} = \delta_{ij})$ corresponds to $G^2(\phi_1^2 + \phi_2^2) = r^2$ and $A(r)G = 1$.

The dual Riemann tensor and the non-Abelian field strength tensor can be obtained from a tedious but straightforward computation. Clearly both of them are of the same form.

$$\begin{aligned} R^a{}_{ij} \;&=\; \frac{1}{r^2}\epsilon_{ijb}\,\hat{x}^a\hat{x}^b\,(\psi_1^2 + \psi_2^2 - 1) + \frac{1}{r}(\epsilon^a{}_{ij} - \epsilon_{ijb}\hat{x}^a\hat{x}^b)(\psi_1' + B\psi_2) \\ &\quad + (\delta^a{}_j\hat{x}_i - \delta^a{}_i\hat{x}_j)\frac{1}{r}(\psi_2' - B\psi_1) \end{aligned} \tag{12}$$

$$F^a{}_{ij} \;=\; \frac{1}{r^2}\epsilon_{ijb}\,\hat{x}^a\hat{x}^b\,(\varphi_1^2 + \varphi_2^2 - 1) + \frac{1}{r}(\epsilon^a{}_{ij} - \epsilon_{ijb}\hat{x}^a\hat{x}^b)(\varphi_1' + D\varphi_2)$$

$$+(\delta^a{}_j \hat{x}_i - \delta^a{}_i \hat{x}_j)\frac{1}{r}(\varphi_2' - D\varphi_1) \tag{13}$$

Before we write down the reduced form of the action let us denote the determinant of the metric

$$\det e = \sqrt{|g|} = \frac{G^3}{r^2}|A|(\phi_1^2 + \phi_2^2) \tag{14}$$

The actions reduce to the following one dimensional forms (here the repeated greek indices take values of $(1,2)$ and a summation is implied). The Einstein-Hilbert action is

$$S_G = -\int_0^\infty dr \left\{ \psi_\alpha' \epsilon_{\alpha\beta}\phi_\beta + B\psi_\alpha\phi_\alpha + \frac{A}{2}(\psi_\alpha\psi_\alpha + \lambda G^2\phi_\alpha\phi_\alpha - 1)\right\} \tag{15}$$

The Chern-Simons action is

$$\begin{aligned} S_{CS} = \quad & -i\frac{k}{G}\int_0^\infty dr \left\{ \psi_\alpha' \epsilon_{\alpha\beta}\psi_\beta + \psi_2' + B(\psi_\alpha\psi_\alpha + \lambda G^2\phi_\alpha\phi_\alpha - 1) \right. \\ & \left. +\lambda G^2(\phi_\alpha'\epsilon_{\alpha\beta}\phi_\beta + 2A\phi_\alpha\psi_\alpha)\right\} \end{aligned} \tag{16}$$

$$\begin{aligned} S_{YM} = -\frac{2\pi}{e^2 G}\int_0^\infty dr \frac{1}{|A|\phi_\delta\phi_\delta} & \left\{ A^2(\varphi_\alpha\varphi_\alpha - 1)^2 + 2\phi_\gamma\phi_\gamma(\varphi_\beta'\varphi_\beta' \right. \\ & \left. +2D\epsilon_{\alpha\beta}\varphi_\alpha'\varphi_\beta + D^2\varphi_\alpha\varphi_\alpha)\right\} \end{aligned} \tag{17}$$

In the BPS limit ($\nu = 0$) and the broken phase ($h_0 \neq 0$) the Higgs term is

$$S_H = -\frac{2\pi G}{e^2}\int_0^\infty dr \left\{ \frac{1}{|A|}\phi_\alpha\phi_\alpha h'^2 + 2h^2|A|\varphi_\alpha\varphi_\alpha \right\} \tag{18}$$

From here we will assume that A(r) is positive so that we may drop the absolute value sign. We will see that this requirement is satisfied in our solution.

We are interested both in the singular and the non-singular solutions. For the case of finite action and non-singular solutions the boundary conditions for the gauge and Higgs sector follow as

$$\varphi_1(0) = 1, \quad \varphi_2(0) = 0 \qquad \varphi_1(\infty) = \varphi_2(\infty) = 0 \tag{19}$$

$$h(0) = 0 \qquad h(\infty) = h_0 \qquad D(\infty) = 0 \tag{20}$$

The equations of motion of the full theory are

$$\delta B: \qquad \psi_\alpha\phi_\alpha + i\frac{k}{G}(\psi_\alpha\psi_\alpha + \lambda G^2\phi_\alpha\phi_\alpha - 1) = 0 \tag{21}$$

$$\delta\psi: \qquad \epsilon_{\alpha\beta}\phi'_\beta - B\phi_\alpha - A\psi_\alpha + i\frac{k}{G}(2\epsilon_{\alpha\beta}\psi'_\alpha + 2B\psi_\beta + 2A\lambda G^2\phi_\beta) = 0 \tag{22}$$

$$\delta D: \qquad \epsilon_{\alpha\beta}\varphi'_\alpha\varphi_\beta + D\varphi_\alpha\varphi_\alpha = 0 \tag{23}$$

$$\delta h: \qquad \left\{\frac{\phi_\alpha\phi_\alpha h'}{A}\right\}' - 2hA\varphi_\alpha\varphi_\alpha = 0 \tag{24}$$

$$\begin{aligned}\delta\phi: \qquad & \epsilon_{\alpha\beta}\psi'_\alpha + B\psi_\beta + A\lambda G^2\phi_\beta - i\frac{k}{G}\lambda G^2(2\epsilon_{\alpha\beta}\phi'_\alpha - 2B\phi_\beta - 2A\psi_\beta) \\ & -\frac{4\pi}{e^2G}\frac{\phi_\beta A}{(\phi_\gamma\phi_\gamma)^2}(\varphi_\alpha\varphi_\alpha - 1)^2 + \frac{4\pi G}{e^2}\frac{\phi_\beta h'^2}{A} = 0\end{aligned} \tag{25}$$

$$\delta\varphi: \quad \left\{\frac{\varphi'_\alpha + D\epsilon_{\alpha\beta}\varphi_\beta}{A}\right\}' - \frac{\varphi_\alpha A}{\phi_\gamma\phi_\gamma}(\varphi_\beta\varphi_\beta - 1) - \frac{D}{A}(D\varphi_\alpha + \epsilon_{\beta\alpha}\varphi'_\beta) - G^2h^2A\varphi_\alpha = 0 \tag{26}$$

$$\begin{aligned}\delta A: \qquad & \psi_\alpha\psi_\alpha + \lambda G^2\phi_\alpha\phi_\alpha - 1 + 4ik\lambda G\phi_\alpha\psi_\alpha + \frac{8\pi G}{e^2}h^2\varphi_\alpha\varphi_\alpha + \frac{4\pi}{Ge^2}\frac{(\varphi_\alpha\varphi_\alpha - 1)^2}{\phi_\gamma\phi_\gamma} \\ & -\frac{4\pi}{Ge^2}\frac{1}{A^2}\left\{2\varphi'_\alpha\varphi'_\alpha + 4D\varphi'_\alpha\epsilon_{\alpha\beta}\varphi_\beta + 2D^2\varphi_\alpha\varphi_\alpha + G^2h'^2\phi_\alpha\phi_\alpha\right\} = 0\end{aligned} \tag{27}$$

In general, because of the Chern-Simons term, solutions to the equations of motion will be complex. Complex dreibein and spin connection, however, will change the geometry

drastically. For example, the notion of a positive definite metric will be lost. Hence, we restrict ourselves to the real solutions only. This means that the Chern-Simons term decouples from the rest. The equations of motion for the Chern-Simons gravity are exactly the equations one gets for Einstein-Hilbert gravity without the matter fields. This fact is no secret because we know that at the classical level Einstein-Hilbert theory is equivalent to Chern-Simons theory of gravity. In this way we have obtained a nice system where we can try to analyze the effect of gravity on three dimensional 't Hooft-Polyakov Instantons. It is clear that gravity itself is not disturbed by the instantons because of the Chern-Simons term. A similar situation arises in four dimensional Euclidean Einstein-Yang-Mills theory [11]. Charap and Duff showed that in the $4D$ theory gauge theory instantons, having a vanishing energy momentum tensor, do not disturb the geometry. But the effect of gravity on the instantons is quite drastic. Using these facts we take on the job of obtaining solutions to the equations in the next section.

3 Solutions of the Equations of Motion

As already stated, we are only looking for real solutions, hence the Chern-Simons term yields the following equations:

$$\epsilon_{\alpha\beta}\phi'_\beta - A\psi_\alpha - B\phi_\alpha = 0 \tag{28}$$

$$\epsilon_{\alpha\beta}\psi'_\beta - B\psi_\alpha - \lambda G^2 A\phi_\alpha = 0 \tag{29}$$

$$\psi_\alpha\psi_\alpha + \lambda G^2\phi_\alpha\phi_\alpha - 1 = 0 \tag{30}$$

$$\phi_\alpha\psi_\alpha = 0 \tag{31}$$

The general solutions of these equations, compatible with the regularity conditions at the origin, were given in [12]

$$\psi_1 = \frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\cos\Omega(r) \qquad \psi_2 = \frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\sin\Omega(r) \tag{32}$$

$$\phi_1 = f(r)\frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\sin\Omega(r) \qquad \phi_2 = -f(r)\frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\cos\Omega(r) \tag{33}$$

$$A = -\frac{f(r)'}{1+\lambda G^2 f^2(r)} \qquad B = \Omega'(r) \tag{34}$$

$f(r)$ and $\Omega(r)$ are arbitrary functions at this point. At the level of the classical equations of motions one can pick any functions. When we compute the actions for the gravity sector we will see that their boundary values, namely $f(0)$, $f(\infty)$, $\Omega(0)$, $\Omega(\infty)$ are of extreme importance for the quantum theory.

We have postponed the issue of gauge fixing up until now. We need to see "how arbitrary" $f(r)$ and $\Omega(r)$ are and whether we can gauge-fix any of them. A look at the equations of motion will reveal that there is a remaining $U(1)$ symmetry which is not broken by the instanton ansatz. Under this symmetry the fields transform in the following way

$$
\begin{aligned}
\tilde{\phi}_1 &= \phi_1 \cos\theta(r) + \phi_2 \sin\theta(r) \\
\tilde{\phi}_2 &= -\phi_1 \sin\theta(r) + \phi_2 \cos\theta(r) \\
\tilde{\psi}_1 &= \psi_1 \cos\theta(r) + \psi_2 \sin\theta(r) \\
\tilde{\psi}_2 &= -\psi_1 \sin\theta(r) + \psi_2 \cos\theta(r) \\
\tilde{B} &= B - \theta'(r) \qquad \tilde{A} = A
\end{aligned} \tag{35}
$$

So $f(r)$ is intact under this symmetry but $\Omega(r)$ is transformed. [3] $\theta(r)$ is the gauge parameter. Choosing $\theta(r) = \Omega(r)$ one can work with the following gauge equivalent fields

$$
\begin{aligned}
\tilde{\psi}_1 &= \frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\,, & \tilde{\psi}_2 &= 0 \\
\tilde{\phi}_2 &= -f(r)\frac{1}{\sqrt{1+\lambda G^2 f^2(r)}}\,, & \tilde{\phi}_1 &= 0 \\
A &= -\frac{f(r)'}{1+\lambda G^2 f^2(r)}\,, & \tilde{B} &= 0
\end{aligned} \tag{36}
$$

The line element in the polar coordinates takes the following form.

$$
(ds)^2 = \frac{G^2}{1+\lambda G^2 f^2}\left\{ f^2 d\Omega_2 + \frac{1}{1+\lambda G^2 f^2}(df)^2 \right\} \tag{37}
$$

[3] All the actions except the Chern-Simons term is invariant under these transformations. Chern-Simons term transforms like $\delta S_{CS} = -\frac{ik}{G}(\gamma(\infty) - \gamma(0))$. As a compact subgroup of $SO(3)$ the remaining $U(1)$ can be parameterized $g(\vec{x}) = e^{i\gamma(r)x^i\sigma^i}$. Where σ^i are the Pauli matrices. If one is working in a compact space, like S^3, for the gauge invariance of the path integral k/G will be quantized. This is because $(\gamma(\infty) - \gamma(0))$ is the winding number of $g(\vec{x})$. On the other hand in an open ball like the one we deal with in this paper the Chern-Simons coefficient is not quantized. In this case $(\gamma(\infty) - \gamma(0))$ becomes a collective coordinate which should be summed over in the path integral. See the discussion in [13].

Once again we should emphasize that the spaces that are described by this metric are constant curvature spaces which satisfy $R^a_{ij} = -\lambda\epsilon^a_{bc}e^b_i e^c_j$ and $R = -6\lambda$. These are local properties of the space times. In the quantum theory global properties of the space time are needed. Below we will show that the above metric describes many global-y inequivalent space-times depending on the choice of the boundary values of $f(r)$. In terms of the gauge theory language: the space of $f(r)$ functions have a non-trivial topology. Before we start the discussion of the actions in the gravity sector let us find the simplified equations of the gauge sector.

Using the solutions of the Chern-Simons equations of motion to simplify the equations for the Higgs and Yang-Mills fields, we can see that the resulting relations are still somewhat complicated. To see the solution more clearly one can make certain choices of gauges. For example a look at the action will reveal that the unbroken $U(1)$ acts in a way that keeps the following complex function invariant

$$\eta(r) = (\varphi_1 + i\varphi_2)e^{-i\int^r D(r')dr'} \tag{38}$$

If the Yang-Mills action is written in terms of $\eta(r)$ obviously none of the functions $(\varphi_1, \varphi_2, D)$ will appear in the action. So we can choose a gauge (the singular gauge) in which $\varphi_2 = D = 0$. Denoting $\varphi_1 = \varphi$, the remaining *independent* equations read as

$$h' = -\frac{A}{G(\phi_\alpha\phi_\alpha)}(\varphi^2 - 1) \tag{39}$$

$$h = -\frac{1}{GA}\frac{\varphi'}{\varphi} \tag{40}$$

These are the Bogomol'nyi type first order equations for the gravitating instanton. These equations reduce to the well known exactly solvable equations in the flat space limit. Writing the above equations explicitly in terms of the solutions of the gravity part one obtains

$$h' = -\frac{|f'(r)|}{Gf(r)^2}(\varphi^2 - 1) \tag{41}$$

$$h = -\frac{\left(1 + \lambda G^2 f(r)^2\right)}{G|f'(r)|}\frac{\varphi'}{\varphi} \tag{42}$$

The coordinate function $f(r)$ explicitly enters in the equations so one should make sure that the existence of the solutions does not depend on the local coordinates. But the global properties of the space time will be important as it should be expected. For generic $f(r)$ there are exact solutions which will be depicted in the next section. But these are all infinite action. Finite actions solutions will be found numerically. But first one needs to make a choice of $f(r)$

We write the gauge sector action in the following form

$$\begin{aligned} S_{YM} + S_H = \quad & -\frac{4\pi}{e^2 G}\int dr \Big\{ \frac{1}{A}\Big(\varphi' + GAh\varphi\Big)^2 - 2G\varphi'\varphi h \\ & + \frac{G^2}{2A}\phi_\alpha\phi_\alpha\Big(h' + \frac{A}{G\phi_\gamma\phi_\gamma(\varphi^2-1)}\Big)^2 - Gh'(\varphi^2-1)\Big\} \end{aligned} \tag{43}$$

If the equations of motion are satisfied the integrand becomes a full derivative and after integration one obtains

$$S_{Instanton} = -\frac{4\pi h(\infty)}{e^2} \tag{44}$$

The result does not explicitly dependent on the cosmological constant. It, however, can be seen from the numerical solutions that $h(\infty)$ depends on the cosmological constant.

Einstein-Hilbert action can be computed to be

$$S_G = -\lambda G^2 \int_0^\infty dr \frac{|f'(r)|f(r)^2}{(1+\lambda G^2 f(r)^2)^2} \tag{45}$$

Observe that the integrand is a total derivative which can integrated to give

$$S_G = \frac{1}{2G\sqrt{\lambda}}\Big\{\frac{-G\sqrt{\lambda}|f(\infty)|}{1+G^2\lambda f^2(\infty)} + \frac{G\sqrt{\lambda}|f(0)|}{1+G^2\lambda f^2(0)} + \arctan\big[G\sqrt{\lambda}|f(\infty)|\big] - \arctan\big[G\sqrt{\lambda}|f(0)|\big]\Big\} \tag{46}$$

It is clear that homotopically inequivalent $f(r)$'s characterize different spaces. In what follows we will work on two different spaces. The first one is given by $f(r) = -r/G$. The gravitational action reads as [4]

$$S_G = -\frac{\pi}{4\sqrt{\lambda}G} \tag{47}$$

[4] The Chern-Simons action will be $S_{CS} = -\frac{ik}{G}(\Omega(\infty) - \Omega(0))$ which is exactly equal to the gauge non-invariant part.

This solution is a gravitational instanton (of course it is not self-dual) and it is not a gauge copy of the AdS space which has a zero action. The trivial vacuum AdS solution is

$$Gf(r) = -r/(1 - \frac{\lambda}{4}r^2) \tag{48}$$

which has a zero action.

4 Numerical Computations

For the gravitational instanton solution where $Gf(r) = -r$ the curved space BPS equations become

$$h'(r) = -\frac{1}{r^2}(\varphi^2(r) - 1) \tag{49}$$

$$\varphi'(r) = -\frac{1}{(1 + \lambda r^2)}h(r)\varphi(r) \tag{50}$$

and the line element becomes

$$(ds)^2 = \frac{1}{1 + \frac{\lambda}{4}r^2}\left\{r^2 d\Omega_2 + \frac{(dr)^2}{1 + \lambda r^2}\right\} \tag{51}$$

In the flat space limit ($\lambda = 0$) one has the well-known BPS solution

$$\varphi(r) = \frac{h_0 r}{\sinh(h_0 r)}; \qquad h(r) = -\frac{1}{r} + h_0 \coth(h_0 r) \tag{52}$$

where $h(\infty) = h_0$. For non-zero λ the solutions can be obtained numerically and they are plotted in figure 1 and figure 2. For any positive value of λ there is a solution. Non-zero λ solutions take values between the BPS instanton solution, (52), and the trivial vacuum solution ($h(r) = 0, \varphi(r) = 1$). For very large values of λ the solution approaches the trivial vacuum solution. For negative values of λ (i.e. the de Sitter case with our conventions) the existence of the horizon introduces singularities and there are no finite action solutions.

As another example of a coordinate choice, let $Gf(r) = -r/(1 - \frac{\lambda}{4}r^2)$ which gives the AdS space. Then the curved BPS equations become:

$$h'(r) = -\frac{(1 + \frac{\lambda}{4}r^2)}{r^2}(\varphi^2(r) - 1) \tag{53}$$

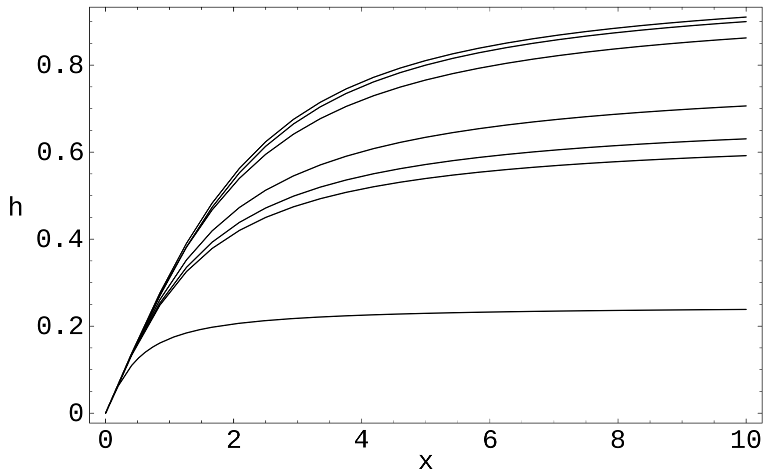

Figure 1: Higgs field $h(x)$ is shown for various values of λ in the choice $Gf = -r$. $h(\infty)$ approaches to zero (vacuum solution) if λ is increased. The above values, starting from the top correspond to $\lambda = 0$, 4.10^{-3} ,$\lambda = 0.1$, $\lambda = 0.5$, $\lambda = 0.8$, $\lambda = 1$ and $\lambda = 10$ in units of h_0^2.

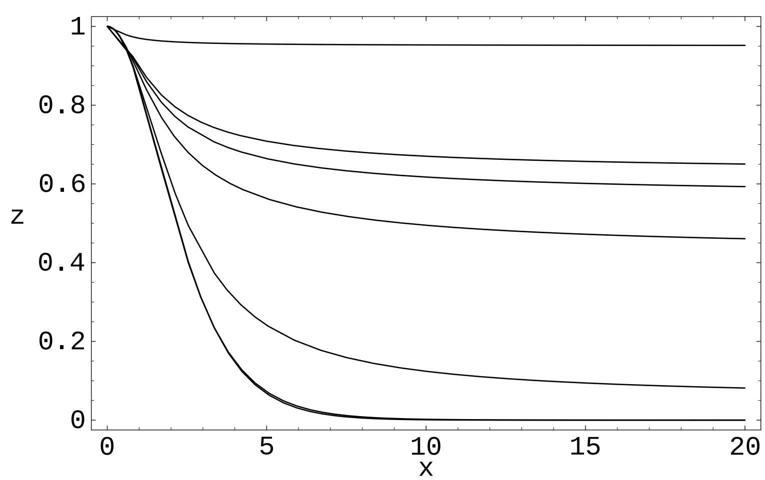

Figure 2: Non-zero component of the gauge field $z(x)$ is shown for various values of λ in the choice $Gf = -r$. $z(\infty)$ approaches to 1 (vacuum solution) if λ is increased. The above values, starting from the bottom correspond to $\lambda = 0, 4.10^{-3}$,$\lambda = 0.1$, $\lambda = 0.5$, $\lambda = 0.8$, $\lambda = 1$ and $\lambda = 10$ in units of h_0^2.

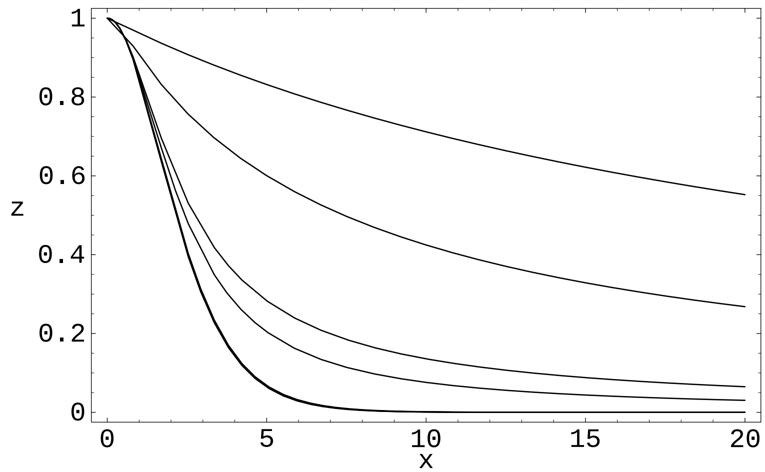

Figure 3: Non-zero component of the gauge field $z(x)$ is shown for various values of λ for the choice $Gf(r) = -r/(1 - \frac{\lambda}{4}r^2)$. The above values, starting from the bottom correspond to $\lambda = 0, 4.10^{-3}$,$\lambda = 0.1$, $\lambda = 0.5$, $\lambda = 0.8$, $\lambda = 1$ and $\lambda = 10$ in units of h_0^2.

$$\varphi'(r) = -\frac{1}{(1 + \frac{\lambda}{4}r^2)} h(r)\varphi(r) \tag{54}$$

With this choice of coordinate, the line element becomes

$$(ds)^2 = \frac{1}{(1 + \frac{\lambda}{4}r^2)^2} \left\{ r^2 d\Omega_2 + (dr)^2 \right\} \tag{55}$$

The numerical solutions to these equations are shown in 3 and 4. The gauge field behaves more or like the previous case but it approaches rather slowly to the vacuum solution, $(h(r) = 0, \varphi(r) = 1)$, when the cosmological constant is increased. The Higgs field does not approach to the vacuum solution and it diverges when the cosmological constant is increased.

5 Singular Solutions

In this section we would like to point out an exact solution to equations of motion . These solutions are singular and have no non-trivial limits in the flat space.

$$\varphi(r) = -\sqrt{\lambda} G f(r), \qquad h(r) = \frac{1 + \lambda G^2 f(r)^2}{f(r) G} \tag{56}$$

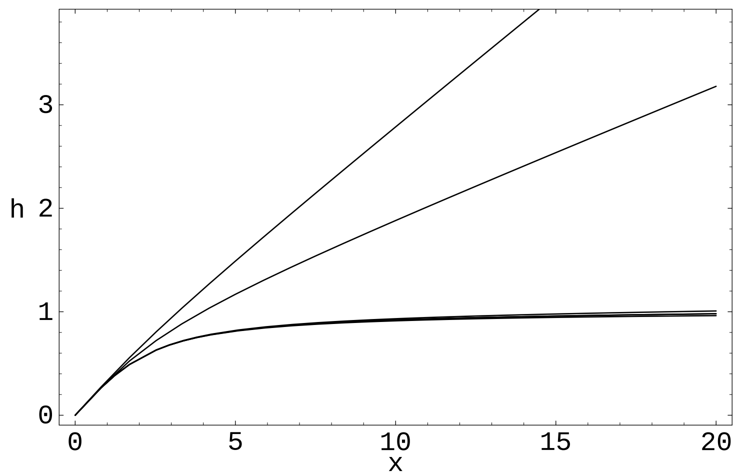

Figure 4: Higgs field $h(x)$ is shown for various values of λ in the choice $Gf(r) = -r/(1 - \frac{\lambda}{4}r^2)$. $h(\infty)$ approaches grows unbounded if λ is increased. The above values, starting from the top correspond to $\lambda = 0$, 4.10^{-3} ,$\lambda = 0.1$, $\lambda = 0.5$, $\lambda = 0.8$, $\lambda = 1$ and $\lambda = 10$ in units of h_0^2.

For any choice of $f(r)$ it is not possible to meet the finite action conditions for the gauge theory instantons. So these solutions are singular.

For definiteness let us rewrite these solutions in the both coordinate examples chosen above. For $Gf = -r$ we have

$$\varphi(r) = \sqrt{\lambda}r, \qquad h(r) = -(\frac{1}{r} + \lambda r) \tag{57}$$

For the choice $Gf(r) = -r/(1 - \frac{\lambda}{4}r^2)$

$$\varphi(r) = \sqrt{\lambda}\frac{r}{(1 - \frac{\lambda}{4}r^2)}, \qquad h(r) = -\frac{(1 + \frac{\lambda}{4}r^2)^2}{r(1 - \frac{\lambda}{4}r^2)} \tag{58}$$

These solutions are not gauge copies of the trivial vacuum solutions

6 Conclusion

We have shown that when three dimensional Euclidean gravity is coupled to Yang-Mills and Higgs fields the equations of motion reduce to first order equations of the Bogomol'nyi type. We found singular and regular solutions. Our main result is that, if the three dimensional space is a gravitational instanton, there are finite action solutions for any

positive semi-definite value of the cosmological constant. Depending on the numerical value of the cosmological constant these solutions take values between the BPS solution and the trivial vacuum solution. The action can be calculated exactly and is given by (44) and the gravitational instanton action is (47). Finite actions solutions are stable and one can define a topological charge which is the magnetic charge. This can be done following 't Hooft's definition of an Abelian field strength outside the instanton core.

We have also showed that there if the cosmological constant is small ($\lambda \leq 0.8 h_0^2$ see fig 3 and 4) there are finite action solutions in the AdS space. The space itself has zero action.

In addition to the Yang-Mills term, if a Chern-Simons term is added for the gauge sector one should look at the complex gauge configurations as it was pointed out in [13]. In this case one cannot use a singular gauge as the Chern-Simons term trivially vanishes. The theory will have the following additional action

$$S = -\frac{i\kappa}{e^2} \int d^3x \epsilon^{\mu\nu\lambda} \mathrm{tr}\Big(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \Big) \tag{59}$$

Using the symmetric ansatz one gets

$$S = \frac{4i\pi\kappa}{e^2} \int_0^\infty \Big[\epsilon_{\alpha\beta} \varphi'_\alpha \varphi_\beta + D(\varphi_\alpha \varphi_\alpha - 1) - \varphi'_2 \Big] \tag{60}$$

Clearly this term vanishes for the singular gauge which is too restrictive. In some other gauges (i.e $D = 0$ and $\varphi_2 \neq 0$) we expect complex solutions. For the case of Einstein-Maxwell-Chern-Simons theory with a Lorentzian metric we refer the reader to [14, 15].

In the context of a non-supersymmetric theory (like the one we dealt with in this paper) h_0 and λ are given to define the theory. So one changes the theory if these parameters are changed at the classical level. So our results mainly mean that there are finite action instanton solutions for those theories which have suitable pairs of λ and h_0. On the other hand if our theory is considered as a bosonic part of a Supergravity theory where a moduli space (for the Higgs field) exists then for given λ there are many solutions.

7 Acknowledgements

The research of B. T. is supported by PPARC Grant PPA/G/O/1998/00567. The work of A. F. was supported in part by DOE grant DE-FG02-94ER-40823.

References

[1] S. Deser ,R. Jackiw and G. 't Hooft ,*Ann. Phys. (N.Y.)* **152** (1984) 220 ; G. 't Hooft ,*Comm. Math. Phys.* **117** (1988) 685 ;S. Deser and R. Jackiw ,*Comm. Math. Phys.* **118** (1988) 495; S. Carlip "Quantum Gravity in 2+1 Dimensions" Cambridge University Press (1998)

[2] M. Bańados, C. Teitelboim and J. Zanelli, *Phys. Rev. Lett.* **69** (1992) 1849

[3] C. Martinez, C. Teitelboim and J. Zanelli, "Charged Rotating Black Hole in Three Space-Time Dimensions", preprint hep-th/9912259

[4] R. Bartnik and J. McKinnon, *Phys. Rev. Lett.* **61** (1988) 141

[5] M. S. Volkov and D. V. Gal'tsov "Gravitating Non-Abelian Solitions and Black Holes with Yang-Mills Fields", preprint hep-th/9810070

[6] Y. Brihaye, B. Hartmann and J. Kunz, *Phys. Lett.* B**441** (1998) 77; E. Winstanley, *Class. Quant. Grav.***16** (1999) 1963; A. R. Lugo and F. A. Schaposnik, *Phys. Lett.* B**467** (1999) 43; J. Bjoraker and Y. Hosotani, *Phys. Rev. Lett.* **84** (2000) 1853

[7] S. Deser *Class. Quant. Grav.***1** (1984) L1

[8] B. R. Greene, S. C. Mathur and C. M. O'Neill *Phys. Rev.* D**47** (1993) 2242

[9] A. Achucárro and P.K. Townsend, *Phys. Lett.* B**180** (1986) 89

[10] E. Witten, *Nucl. Phys.* B**311** (1988) 46

[11] J. M. Charap and M. J. Duff, *Phys. Lett.* B**69** (1977) 445

[12] B. Tekin, "On the Relevance of Singular Solutions in dS_3 and AdS_3 Gravity", hep-th/9902090

[13] B. Tekin, K. Saririan and Y. Hosotani *Nucl. Phys.* B**539** (1998) 720

[14] S. Fernando and F. Mansouri, "Rotating Charged Solutions to Einstein Maxwell Chern-Simons Theory in 2+1 Dimensions", gr-qc/9705016

[15] T. Dereli and Yu. N. Obukhov, "General Analysis of Self-dual Solutions for the Einstein-Maxwell-Chern-Simons Theory in 1+2 Dimensions", gr-qc/0001017